Evolution of Cosmological Perturbations in the Presence of Primordial Magnetic Fields

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Possible existence of the primordial magnetic fields has affected the structure formation of the universe. In this paper it is shown that the initial conditions for density perturbations with magnetic fields derived in previous works are inconsistent with Einstein equations. We find that this inconsistency arises due to the unwanted cancellation of contributions from the magnetic fields and primordial radiations. A complete set of equations and consistent initial conditions in the long wavelength limit are given with an explicit derivation in the covariant approach with CDM frame, by newly taking into account a non-relativistic matter contribution in the radiation dominated era. By solving these equations numerically, we derive the angular spectrum of cosmic microwave background anisotropies and the matter power spectrum with magnetic fields. We find that the amplitude of the angular power spectrum of CMB anisotropies can alter at most a order of magnitude at $l \lesssim 4000$ compared with the previous results in the literature.

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I. INTRODUCTION

There are mounting evidences that the large scale magnetic fields are present in various objects in the universe. Not only galaxies, but also clusters of galaxies contain their own magnetic fields with the field strength of $\sim 10^{-6}$ Gauss and the coherence length of $1-10$ kpc (for a review, [1]). Furthermore, there have been some observations which indicate that they exist even in larger scales, such as in superclusters [2].

Yet the origin of such large scale magnetic fields is still a matter of debate. Magnetic fields in spiral galaxies are assumed to be continuously generated and maintained by dynamo mechanism [3]. However, one still needs to explain the origin of seed fields necessary for dynamo action to take place. Astrophysical origins of such seed fields, often involving stellar activities or the Biermann battery in non-adiabatic processes [4, 5, 6], may explain the strength and the total amount of magnetic fields with help from the dynamo mechanism. Their coherence scales are, however, much smaller than those of intergalactic magnetic fields and thus magnetic fields generated from these mechanisms could not be directly the origin of large-scale magnetic fields.

On the other hand, primordial origins, often related with inflation [7, 8, 9, 10, 11, 12] or second order effects through cosmological vector modes [13, 14, 15, 16, 17, 18, 19, 20], have no difficulty in accounting for the length of coherence. The observational facts that there exist significant magnetic fields in objects at large redshift may support the hypothesis of primordial origin of the large scale magnetic fields [21, 22]. If this is the case, it is expected that primordial magnetic fields should have formed imprints in the anisotropies of cosmic microwave background (CMB) through their stress energy tensor and their Lorentz force on the baryon-photon fluid before cosmological recombination. Therefore, it is important to develop the cosmological perturbation theory with primordial magnetic fields in order to search for signs of magnetic fields in the observed CMB maps.

In recent years the effects of stochastic primordial magnetic fields on the evolution of cosmological perturbations have been developed independently by several authors (for a review [23]). The scalar type perturbations, which is related to density fluctuations, have been considered by [24, 25, 26, 27, 28, 29, 30]. The vector type perturbations, which would give the most dominant contribution to CMB anisotropies at small angular scales, have been studied by [31, 32, 33, 34, 35, 36, 37], and also the tensor type perturbations [32, 38]. All of these studies suggest that, from the currently available CMB data, the amplitude of primordial stochastic magnetic fields should be at most a few times 10^{-9} Gauss or below at the relevant scales [39].

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In the present paper we reconsider the scalar type cosmological perturbations with primordial magnetic fields. Within the standard cosmological perturbation theory, an initial condition of the perturbation Fourier mode with wavenumber k is set when the mode is well outside the horizon ($k\tau \ll 1$ with τ being conformal time) and when the universe is deep in the radiation dominated era, neglecting non-relativistic matter contributions (for example, see [40]). Following this standard practice the initial conditions of density perturbations with primordial magnetic fields have been derived [29, 41, 42, 43, 44]. We find, however, that this procedure does not give us a consistent initial conditions in the presence of magnetic fields, because of the unwanted cancellation of contributions from the magnetic fields and primordial radiations. This cancellation makes the system unstable and violates the constraint of perturbed Einstein equations to be satisfied. As we shall show below, this difficulty can be removed by considering the significant contributions from the non-relativistic matter at initial conditions.

The paper is organized as follows. In Section II, we set up basic equations for the perturbation theory including primordial magnetic fields. We adopt the covariant approach to derive equations while equations in Refs. [29, 30, 40, 41] are derived in a conventional synchronous gauge. The anisotropic stress and the Lorentz force of the magnetic field are defined and the equation of motion for baryons is derived. We also define the spectrum of the magnetic field. In Section III, we point out an inconsistency in the previous works and derive the initial condition of the purely magnetic mode including the non-relativistic matter contribution. In Section IV we show the numerical calculation of CMB and matter power spectra. Finally, we conclude this work.

II. EQUATIONS

A. Basic equations

Here we set up equations. In what follows we take the covariant approach with CDM frame to eliminate the gauge freedom [45, 46, 47]. In this frame, we define variables on the supersurface orthogonal to the CDM 4-velocity u_μ. Then one can define the anisotropic expansion rate (shear) σ and the inhomogeneous expansion rate $\mathcal{Z}$ from the covariant derivative of u_μ. In the scalar mode, we can neglect vorticity. In addition, we introduce the Weyl tensor, which is the traceless part of the Riemann curvature tensor. The Weyl tensor vanishes in the background FRW spacetime. Since the magnetic part of the Weyl tensor is negligible in the scalar mode, we define the electric part of the Weyl tensor as Φ. Linearizing the Bianchi identities and Ricci identities, we obtain the following equations for Φ, σ and $\mathcal{Z}$. Three propagation equations for Φ, σ and $\mathcal{Z}$ are:

$$\Phi' + \mathcal{H}\Phi + \frac{1}{2k}\kappa\rho a^2\left(\tilde{\gamma}\sigma + R_f q_f\right) + \frac{\mathcal{H}}{2k^2}\kappa\rho a^2(3\tilde{\gamma}-1)R_f\pi_f - \frac{1}{2k^2}\kappa\rho a^2 R_f\pi'_f = 0\ , \tag{1}$$

$$\sigma' + \mathcal{H}\sigma + k\Phi + \frac{1}{2k}\kappa\rho a^2 R_f\pi_f = 0\ , \tag{2}$$

$$\mathcal{Z}' + \mathcal{H}\mathcal{Z} + \frac{1}{2k}\kappa\rho a^2 R_f(\Delta_f + 3\delta P_f) = 0\ . \tag{3}$$

Two constraint equations are:

$$\frac{2}{3}(\mathcal{Z}-\sigma) + \frac{1}{k^2}\kappa\rho a^2 R_f q_f = 0\ , \tag{4}$$

$$2\Phi - \frac{1}{k^2}\kappa\rho a^2(R_f\Delta_f + R_f\pi_f) - \frac{3\mathcal{H}}{k^3}\kappa\rho a^2 R_f q_f = 0\ . \tag{5}$$

The prime " $'$ " denotes the derivative with respect to the conformal time τ and $\mathcal{H} \equiv a'/a$, where a is the scale factor. Here we defined $\tilde{\gamma}$ as $p = (\tilde{\gamma}-1)\rho$, where p is the total pressure and ρ is total energy density, and $\kappa \equiv 8\pi G$. The subscript " $_f$ " means the sum of photon (γ), neutrino (ν), baryon (b), CDM (c) and magnetic field (B). The density fluctuation Δ_f, heat flux q_f and anisotropic stress π_f are normalized with their energy density ρ_f except for the magnetic field. For the magnetic field variables we normalize Δ_B and π_B with the photon energy density ρ_γ because we consider the case that the magnetic field does not contribute to the background spacetime. Then the total energy density ρ is $\rho = \rho_\gamma + \rho_\nu + \rho_b + \rho_c$. Here we define the energy fraction R_f as $R_\gamma = \rho_\gamma/\rho$, $R_\nu = \rho_\nu/\rho$, $R_b\tau = \rho_b/\rho$, $R_c\tau = \rho_c/\rho$ and $R_m \equiv \frac{3}{4}(R_b + R_c)$. In the deep radiation dominated era, the background energy densities of baryon and CDM are negligible. In this epoch, $\rho \simeq \rho_\gamma + \rho_\nu$, $R_\gamma + R_\nu \simeq 1$ and all R_f's are constant. Note that R_b and R_c have dimension of (length)$^{-1}$. Note that in the metric perturbation approach, there are four, not five, equations as shown in Ref. [40]. This is because our variables Φ, σ and $\mathcal{Z}$ are not independent from each other.

Next we introduce the stochastic magnetic field and evolution equations for each component. In this work, we assume that the magnetic field can be treated as the first order quantity and does not contribute to the background

evolution. Then the perturbed Einstein equation is described as

$$\delta G^{\mu}_{\ \nu} = 8\pi G(\delta T^{\mu}_{\ \nu} + T_{B\ \nu}^{\ \mu}) \ . \tag{6}$$

Here $T_{B\ \nu}^{\ \mu}$ is the energy-momentum tensor for the magnetic field:

$$T_{B\ 0}^{\ 0}(x) = -\frac{B^2(x)}{8\pi a^4} \ , \tag{7}$$

$$T_{B\ j}^{\ i}(x) = \frac{1}{4\pi a^4}\left(\frac{B^2(x)}{2}\delta^i_{\ j} - B^i(x)B_j(x)\right) \ , \tag{8}$$

where $B_i(x)$ is the magnetic field strength at present time and we assume that the conductance of the universe is infinite, i.e. $E_i = 0$. We decompose the space-space part of energy-momentum tensor as

$$T_{B\ j}^{\ i}(x) = \frac{1}{3}\frac{B^2(x)}{8\pi a^4}\delta^i_{\ j} + \frac{1}{4\pi a^4}\left(-B^i(x)B_j(x) + \frac{1}{3}\delta^i_{\ j}B^2(x)\right) \ , \tag{9}$$

where the first term in the r.h.s. is the trace part and second term is the traceless part. The traceless part is the anisotropic stress. In the Fourier space, the energy density and anisotropic stress of the magnetic field are defined as

$$\rho_\gamma \Delta_B = -T_{B\ 0}^{\ 0}(k) = \delta_i^{\ j} T_{B\ j}^{\ i}(k) \ , \tag{10}$$

$$\rho_\gamma \pi_B = -\frac{3}{2}\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^{\ j}\right) T_{B\ j}^{\ i}(k) \ . \tag{11}$$

Since $T_{B\ \nu}^{\ \mu}(k)$ and ρ_γ are proportional to a^{-4}, Δ_B and π_B are constant. Then $T_{B\ j}^{\ i}(k)$ is expressed as

$$T_{B\ j}^{\ i}(k) = \frac{1}{3}\delta^i_{\ j}\rho_\gamma\Delta_B - \left(\hat{k}^i\hat{k}_j - \frac{1}{3}\delta^i_{\ j}\right)\rho_\gamma\pi_B \ . \tag{12}$$

If the magnetic field exists, the time evolution of the baryon fluid is affected by the Lorentz force. The energy conservation for baryon is described as

$$\delta T_{b\ \nu;\mu}^{\ \mu} + T_{B\ \nu;\mu}^{\ \mu} = 0 \ , \tag{13}$$

$$\delta T_{b\ i;\mu}^{\ \mu} = \rho_b(q_{bi}' + \mathcal{H}q_{bi}) + \frac{4}{3}an_e\sigma_T\rho_\gamma\left(q_{bi} - \frac{3}{4}q_{\gamma i}\right) \ , \tag{14}$$

$$T_{B\ i;\mu}^{\ \mu} = \frac{1}{4\pi a^4}(\nabla\times\mathbf{B})\times\mathbf{B} = \frac{1}{4\pi a^4}\left((\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla B^2\right) = -\frac{1}{3}\frac{\nabla_j B^2}{8\pi a^4}\delta^j_{\ i} - \frac{\nabla_j}{4\pi a^4}\left(-B^jB_i + \frac{1}{3}\delta^j_{\ i}B^2\right) \ , \tag{15}$$

where n_e is the number density of free electrons and σ_T is the Thomson cross section. Here we have neglected the baryon pressure. Finally, we obtain the equation of motion for baryon in the Fourier space:

$$q_b' + \mathcal{H}q_b + an_e\sigma_T R\left(q_b - \frac{3}{4}q_\gamma\right) = -\frac{3}{4}kRL \ ,$$

$$L \equiv \frac{1}{3}(-\Delta_B + 2\pi_B) \ , \tag{16}$$

where $R \equiv 4\rho_\gamma/(3\rho_b)$. The Lorentz force term does not vanish after the recombination because residual free electrons and ions still interact with neutral atoms, and thus these particles move together. As we see later, the Lorentz force have a large effect on the growth of curvature perturbation after the recombination. Under the circumstances where the electric field is negligible, the continuity equation for baryon is written in the same manner as the standard one:

$$\Delta_b' + k(\mathcal{Z} + q_b) = 0 \ . \tag{17}$$

Photons are coupled with baryons through Thomson scattering. The zero and first moments of Boltzmann equation for photons are

$$\Delta_\gamma' = -k\left(\frac{4}{3}\mathcal{Z} + q_\gamma\right) \ , \tag{18}$$

$$q_\gamma' = \frac{k}{3}(\Delta_\gamma - 2\pi_\gamma) + an_e\sigma_T\left(\frac{4}{3}q_b - q_\gamma\right) \ . \tag{19}$$

In the early epoch at which one should impose the initial conditions, the anisotropic stress and more higher multipoles of photons are negligible.

Before the recombination, baryons and photons are tightly coupled, so that $q_b \simeq 3q_\gamma/4 \equiv v_{\gamma b}$. In the tight-coupling epoch, Eqs. (19) and (16) are combined to

$$v'_{\gamma b} + \frac{\mathcal{H}}{1+R} v_{\gamma b} = \frac{k}{4}\frac{R}{1+R}(\Delta_\gamma - 3L)\ . \tag{20}$$

Other particle species such as neutrino and CDM are treated as collisionless particles. Since neutrinos are relativistic, their evolution should be followed by solving the collisionless Boltzmann equations:

$$\Delta'_\nu = -k\left(\frac{4}{3}\mathcal{Z} + q_\nu\right)\ , \tag{21}$$

$$q'_\nu = \frac{k}{3}(\Delta_\nu - 2\pi_\nu)\ , \tag{22}$$

$$\pi'_\nu = k\left(\frac{2}{5}q_\nu - \frac{3}{5}G_\nu^{(3)}\right) + \frac{8}{15}k\sigma\ , \tag{23}$$

$$G_\nu^{(3)\prime} = \frac{3}{7}k\pi_\nu\ , \tag{24}$$

where we set $G_\nu^{(l)} = 0$ ($l > 3$). CDM can be treated as a nonrelativistic perfect fluid and does not have velocity in our frame, $q_c = 0$. Its energy perturbation evolves as

$$\Delta'_c + k\mathcal{Z} = 0\ . \tag{25}$$

B. Spectrum of the magnetic field

To calculate CMB anisotropies generated from stochastic primordial magnetic fields, we need to specify the spectrum of them. It is shown in Refs. [32, 48] that the magnetic field is damped in small scales $k > k_D$, where k_D is the wavenumber of damping scale. Here we assume that the magnetic field has power-law spectrum in $k < k_D$ in the same manner as in previous works [32, 35, 49],

$$\langle B_i(\mathbf{k})B_j^*(\mathbf{k}')\rangle \ = \ (2\pi)^3\frac{P_{ij}}{2}Ak^{n_B}\delta(\mathbf{k}-\mathbf{k}')\ , \tag{26}$$

with $P_{ij} \equiv \delta_{ij} - \hat{k}_i\hat{k}_j$, which is the divergence free condition of the magnetic field. There are some ways to define the amplitude of the magnetic fields as shown in Ref. [41]. In this work, we define the amplitude of the magnetic field B_λ by smoothing at $\lambda = 1$Mpc with Gaussian window function in Fourier space,

$$B_\lambda^2 \equiv \frac{1}{(2\pi)^3}\int d^3k A k^{n_B}\exp\left(-\lambda^2k^2\right)\ . \tag{27}$$

Integrating Eq. (27), one obtains the two-point correlation of magnetic field,

$$\langle B_i(\mathbf{k})B_j^*(\mathbf{k}')\rangle \ = \ (2\pi)^3 P_{ij}\frac{(2\pi)^{n_B+5}B_\lambda^2}{2\Gamma(\frac{n_B+3}{2})k_\lambda^{n_B+3}}k^{n_B}\delta(\mathbf{k}-\mathbf{k}')\ , \tag{28}$$

Since the energy momentum tensor for magnetic field is quadratic in B, one needs to calculate a convolution in order to obtain the spectrum of Δ_B and L. Although many previous works used approximated spectra, Ref. [30] obtained the exact spectra for several values of n_B. For example, the spectrum for $n_B = -2.5$ is:

$$\begin{aligned} k^3\frac{\mid \Delta_B(k)\mid^2_{n_B=-2.5}}{2\pi^2} &\simeq \frac{17k}{800\pi^6}\left[\frac{(2\pi)^{n_B+5}B_\lambda^2}{2\Gamma(\frac{n_B+3}{2})k_\lambda^{n_B+3}(a^4\rho_\gamma)}\right]^2\ , \\ \mid L(k)\mid^2_{n_B=-2.5} &\simeq \frac{55}{51}\mid \Delta_B(k)\mid^2_{n_B=-2.5}\ . \end{aligned} \tag{29}$$

This spectrum is valid for scales much larger than damping scale, i.e. $k \ll k_D$. Since k_D is sufficiently large, we use Eq. (29) throughout this work.

III. DERIVING INITIAL CONDITION

In previous works [29, 41, 42] initial conditions are derived neglecting the matter contributions. This is a good approximation for the adiabatic mode in the standard model. However, in the existence of the magnetic fields, the compensation mechanism between the radiation energy perturbation and the magnetic energy density makes the matter contributions not to be negligible. We can understand it as follows. The initial conditions for purely magnetic mode derived in previous works by neglecting matter contributions are [29, 41, 42]

$$\Delta_\gamma = -R_\gamma\Delta_B - \frac{1}{6}(R_\nu\Delta_B - 2\pi_B)k^2\tau^2 \ , \tag{30}$$

$$\Delta_\nu = -R_\gamma\Delta_B + \frac{R_\gamma}{6R_\nu}(R_\nu\Delta_B - 2\pi_B)k^2\tau^2 \ , \tag{31}$$

$$\Delta_b = -\frac{3}{4}R_\gamma\Delta_B + \mathcal{O}(k^2\tau^2) \ , \tag{32}$$

$$\Delta_c = -\frac{3}{4}R_\gamma\Delta_B + \mathcal{O}(k^2\tau^2) \ , \tag{33}$$

$$\mathcal{Z} = \mathcal{O}(k^3\tau^3). \tag{34}$$

In the radiation dominant epoch, if one substitutes these solutions to Eq. (3), one sees that contributions from radiations and magnetic fields compensates each other to obtain

$$\mathcal{Z}' + \mathcal{H}\mathcal{Z} + \frac{1}{2k}\kappa\rho a^2 R_f(\Delta_f + 3\delta P_f) = \frac{3}{2k\tau^2}(-R_m R_\gamma\Delta_B\tau + \mathcal{O}(\tau^3)) + \mathcal{O}(k^3\tau^2) \ , \tag{35}$$

where we neglected the baryon pressure. In the early epoch, $\tau \to 0$, Eq. (35) diverges and does not satisfy Eq. (3). This inconsistency is caused by the matter contribution in r.h.s. of Eq. (35). Thus it is important to derive initial conditions including baryon and CDM perturbations. The situation is similar to those of isocurvature models, in which the metric perturbations at early times are determined by the non-relativistic matter contributions [50].

In order to obtain the appropriate initial conditions with magnetic fields for Φ, σ and $\mathcal{Z}$ up to the leading order in $k\tau$, let us first combine Eqs. (1)-(5). Although the matter contributions make equations complicated, we obtain the following three equations in the radiation dominant epoch: a second-order equation for Φ,

$$\begin{aligned}
&(1+R_m\tau)\Phi'' + (4+3R_m\tau)\mathcal{H}\Phi' + \frac{1}{3}\Big(k^2 - (2\mathcal{H}' + \mathcal{H}^2)R_m\tau\Big)\Phi + \mathcal{O}(\mathcal{H})\sigma \\
&= \frac{3}{2k^2}R_f\pi_f\Big(-2\mathcal{H}^4 + 2\mathcal{H}^2\mathcal{H}' + 2\mathcal{H}\mathcal{H}'' + 2\mathcal{H}'^2 + (-\mathcal{H}^4 + 2\mathcal{H}'^2 + 2\mathcal{H}\mathcal{H}'')R_m\tau\Big) \\
&+ \frac{3}{2k^2}R_f\pi_f'\Big(2\mathcal{H}^3 + 4\mathcal{H}\mathcal{H}' + (\mathcal{H}^3 + 4\mathcal{H}\mathcal{H}')R_m\tau\Big) + \frac{3}{2k^2}R_f\pi_f''(1+R_m\tau)\mathcal{H}^2 \\
&+ \frac{\mathcal{H}^2}{2}(3+2R_m\tau)R_f\pi_f - \frac{\mathcal{H}^2}{2}R_m\tau R_\gamma\Delta_B,
\end{aligned} \tag{36}$$

a third-order equation for σ,

$$\begin{aligned}
&(1+R_m\tau)\sigma''' + (5+4R_m\tau)\mathcal{H}\sigma'' + \Big(4\mathcal{H}' + 6\mathcal{H}^2 + (4\mathcal{H}^2 + 4\mathcal{H}')R_m\tau + \frac{k^2}{3}\Big)\sigma' \\
&+ \Big(2\mathcal{H}'' + 6\mathcal{H}\mathcal{H}' + (2\mathcal{H}'' + 4\mathcal{H}\mathcal{H}')R_m\tau + \frac{k^2}{3}\mathcal{H}\Big)\sigma \\
&= -\mathcal{H}^2(2+R_m\tau)kR_f\pi_f + \frac{\mathcal{H}^2}{2}R_m\tau kR_\gamma\Delta_B \\
&- \Big(4\mathcal{H}'^2 + 4\mathcal{H}\mathcal{H}'' + 12\mathcal{H}^2\mathcal{H}' + (4\mathcal{H}'^2 + 4\mathcal{H}\mathcal{H}'' + 8\mathcal{H}^2\mathcal{H}')R_m\tau\Big)\frac{3}{2k}R_f\pi_f \\
&- \Big(8\mathcal{H}\mathcal{H}' + 6\mathcal{H}^3 + (8\mathcal{H}\mathcal{H}' + 4\mathcal{H}^3)R_m\tau\Big)\frac{3}{2k}R_f\pi_f' \\
&- 2(1+R_m\tau)\mathcal{H}^2\frac{3}{2k}R_f\pi_f'',
\end{aligned} \tag{37}$$

and a first-order equation for $\mathcal{Z}$, σ and Φ,

$$\mathcal{Z}' + \frac{3+2R_m\tau}{1+R_m\tau}\mathcal{H}\mathcal{Z} + \frac{2+R_m\tau}{1+R_m\tau}k\Phi - \frac{2+R_m\tau}{1+R_m\tau}\mathcal{H}\sigma = \frac{3\mathcal{H}^2}{2k}\frac{2+R_m\tau}{1+R_m\tau}R_f\pi_f - \frac{3\mathcal{H}^2}{2k}\frac{R_m\tau}{1+R_m\tau}R_\gamma\Delta_B \ , \tag{38}$$

where we have neglected the baryon pressure and used the adiabatic condition, $\Delta_\gamma = \Delta_\nu = \frac{4}{3}\Delta_b = \frac{4}{3}\Delta_c$. As we have already pointed out, Δ_B and π_B are constant. The magnetic mode is a particular solution of the linearized Einstein equations, while the standard adiabatic and isocurvature modes are the general solutions of them.

In the purely magnetic mode, photons and neutrinos compensate the energy perturbation of the magnetic field initially, i.e. $\Delta_\gamma = \Delta_\nu = -R_\gamma\Delta_B+$(higher order terms) [41, 42, 43, 44], and anisotropic stress of neutrinos compensates that of the magnetic field, i.e. $\pi_\nu = -\frac{R_\gamma}{R_\nu}\pi_B + \pi^{(2)}k^2\tau^2$, where $\pi^{(2)}$ denotes the coefficient of the $\mathcal{O}(k^2\tau^2)$ term. These compensation mechanism and Eqs.(36)-(38) make it possible to derive Φ, σ and $\mathcal{Z}$ up to the second order:

$$\Phi = \frac{9}{2}R_\nu\pi^{(2)} - \frac{R_m}{8k}R_\gamma\Delta_B k\tau \ , \tag{39}$$

$$\sigma = -3R_\nu\pi^{(2)}k\tau + \frac{R_m}{24k}R_\gamma\Delta_B k^2\tau^2 \ , \tag{40}$$

$$\mathcal{Z} = -\frac{R_m}{2k}R_\gamma\Delta_B + \Big(-3R_\nu\pi^{(2)} + \frac{1}{8}\Big(\frac{R_m}{k}\Big)^2 R_\gamma\Delta_B\Big)k\tau \ , \tag{41}$$

where we take into account the matter contribution to the Friedmann equation, namely, $a \simeq \sqrt{\frac{\kappa(\rho_{\nu 0}+\rho_{\gamma 0})}{3}}\tau + \frac{\kappa(\rho_{b0}+\rho_{c0})}{12}\tau^2$ with ρ_{i0} being the energy density of species i at present, and $\mathcal{H} \simeq \tau^{-1} + \frac{1}{3}R_m$ in the radiation dominant epoch [35, 50].

In order to know initial conditions for each component, we need to solve Eqs. (16)-(25). The leading order terms of the neutrino perturbation are determined from the compensation mechanism. From Eqs. (41) and (40), we obtain the neutrino perturbation up to the second order:

$$\Delta_\nu = -R_\gamma\Delta_B + \frac{2}{3}R_m R_\gamma\Delta_B\tau \ , \tag{42}$$

$$q_\nu = -\frac{1}{3}\left(R_\gamma\Delta_B - 2\frac{R_\gamma}{R_\nu}\pi_B\right)k\tau + \frac{1}{9}R_m R_\gamma\Delta_B k\tau^2 \ , \tag{43}$$

$$\pi_\nu = -\frac{R_\gamma}{R_\nu}\pi_B - \left(\frac{2}{15}R_\gamma\Delta_B - \frac{11}{21}\frac{R_\gamma}{R_\nu}\pi_B + \frac{8}{5}R_\nu\pi^{(2)}\right)\frac{1}{2}k^2\tau^2 \ , \tag{44}$$

$$G_\nu^{(3)} = -\frac{3}{7}\frac{R_\gamma}{R_\nu}\pi_B k\tau \ . \tag{45}$$

Again, the definition of $\pi^{(2)}$ is $\pi_\nu = -\frac{R_\gamma}{R_\nu}\pi_B + \pi^{(2)}k^2\tau^2$. Then Eq. (44) leads to

$$\pi^{(2)} = -\frac{1}{42}\frac{R_\gamma}{R_\nu}\frac{14R_\nu\Delta_B - 55\pi_B}{4R_\nu+5} \ . \tag{46}$$

The equation of motion for photon-baryon fluid, Eq. (20), imply that

$$q_\gamma = \frac{1}{3}(R_\nu\Delta_B - 2\pi_B)k\tau + q_\gamma^{(2)}k\tau^2 \ . \tag{47}$$

From Eqs.(4) and (41) and tight coupling approximation, $q_b \simeq 3q_\gamma/4$, we can obtain $q_\gamma^{(2)}$:

$$R_f q_f = \left(\frac{1}{9}R_m R_\nu R_\gamma\Delta_B + R_\gamma q_\gamma^{(2)} + \frac{1}{4}R_b(R_\nu\Delta_B - 2\pi_B)\right)k\tau^2 = \frac{1}{9}R_m R_\gamma\Delta_B k\tau^2 \ ,$$
$$q_\gamma^{(2)} = \frac{1}{9}R_m R_\gamma\Delta_B - \frac{1}{4}\frac{R_b}{R_\gamma}(R_\nu\Delta_B - 2\pi_B). \tag{48}$$

The Boltzmann equations and the solutions of $\mathcal{Z}$, q_ν and q_γ give Δ_ν, Δ_γ and Δ_c up to the third order:

$$\Delta_\nu = -R_\gamma\Delta_B + \frac{2}{3}R_m R_\gamma\Delta_B\tau + \Big(-2\pi^{(2)}R_\nu - \frac{1}{12}\Big(\frac{R_m}{k}\Big)^2 R_\gamma\Delta_B + \frac{1}{6}(R_\gamma\Delta_B - 2\frac{R_\gamma}{R_\nu}\pi_B)\Big)k^2\tau^2 \ , \tag{49}$$

$$\Delta_\gamma = \frac{4}{3}\Delta_b = -R_\gamma\Delta_B + \frac{2}{3}R_m R_\gamma\Delta_B\tau + \Big(-2\pi^{(2)}R_\nu - \frac{1}{12}\Big(\frac{R_m}{k}\Big)^2 R_\gamma\Delta_B - \frac{1}{6}(R_\nu\Delta_B - 2\pi_B)\Big)k^2\tau^2 \ , \tag{50}$$

$$\Delta_c = -\frac{3}{4}R_\gamma\Delta_B + \frac{1}{2}R_m R_\gamma\Delta_B\tau + \Big(-\frac{3}{2}\pi^{(2)}R_\nu - \frac{1}{16}\Big(\frac{R_m}{k}\Big)^2 R_\gamma\Delta_B\Big)k^2\tau^2 \ . \tag{51}$$

In the numerical calculation, we use the curvature perturbation η, $\eta = -(2\Phi + \sigma'/k)$, in place of Φ. Its initial condition is

$$\eta = -6R_\nu \pi^{(2)} + \frac{1}{6}\frac{R_m}{k}R_\gamma \Delta_B k\tau - \frac{1}{48}\left(\frac{R_m}{k}\right)^2 R_\gamma \Delta_B k^2\tau^2 \ . \tag{52}$$

Since this magnetic mode is a particular solution of the linearized Einstein equations, the general solution can be the sum of standard adiabatic mode and purely magnetic mode. In such cases, the total temperature perturbation $\Delta^{\rm tot}$ is represented as

$$\Delta^{\rm tot} = \Delta^{\rm adi} + \Delta^{\rm B} \ . \tag{53}$$

Here $\Delta^{\rm adi}$ is a temperature perturbation from the adiabatic mode, which is calculated with standard adiabatic initial conditions [40], and $\Delta^{\rm B}$ is from the purely magnetic mode. Then the ensemble average is

$$\langle \Delta^{\rm tot}\Delta^{\rm tot*}\rangle = \langle \Delta^{\rm adi}\Delta^{\rm adi*}\rangle + \langle \Delta^{\rm B}\Delta^{\rm B*}\rangle + \langle \Delta^{\rm adi}\Delta^{\rm B*}\rangle + \langle \Delta^{\rm B}\Delta^{\rm adi*}\rangle \ , \tag{54}$$

where the latter two terms are the correlation between the adiabatic and the purely magnetic modes. In what follows we study the three cases, namely, fully correlated case, anti correlated case and uncorrelated case.

IV. RESULT AND DISCUSSION

A. Matter Contribution

The new initial conditions derived in the above are totally different from those used in the previous works. Since we do not neglect matter contribution, Δ's and η have terms of $\mathcal{O}(R_m\tau)$. To see the importance of $R_m\tau$ terms, we calculate the total energy perturbation up to the second order:

$$R_f\Delta_f = -\frac{1}{3}R_m R_\gamma \Delta_B \tau + \left(-2R_\nu \pi^{(2)}k^2 + \frac{7}{12}R_m^2 R_\gamma \Delta_B\right)\tau^2 \tag{55}$$

$$= \left(-\frac{1}{3}R_\gamma \Delta_B + \frac{7}{12}R_m R_\gamma \Delta_B \tau\right)R_m\tau - 2R_\nu\pi^{(2)}k^2\tau^2 \ . \tag{56}$$

We define a wavenumber $k_{\rm mat}$ as the scale in which absolute values of two factors in $\mathcal{O}(\tau^2)$ term in Eq. (55) are equal to each other,

$$\mid 2R_\nu \pi^{(2)}k_{\rm mat}^2 \mid = \left|\frac{7}{12}R_m^2 R_\gamma \Delta_B\right| \ , \tag{57}$$

then we obtain $k_{\rm mat} \simeq 1.2R_m$. If $k < k_{\rm mat}$, $R_f\Delta_f$ is dominated by the matter contribution. Since the wave number $k_{\rm mat}$ correspond to the multipole $l_{\rm mat} \sim k_{\rm mat}\tau_0 \sim 80$, where τ_0 is the present conformal time, we cannot neglect the matter contribution in lower multipoles. When $k > k_{\rm mat}$, we define τ_m as the conformal time when the absolute values of two terms in r.h.s. of Eq.(56) are equal to each other, i.e.

$$k\tau_m \equiv \frac{4R_m R_\gamma \Delta_B k}{36 \mid \pi^{(2)} \mid R_\nu k^2 + 7R_m^2 R_\gamma \Delta_B} \simeq \left(2\frac{k}{R_m} + 1.75\frac{R_m}{k}\right)^{-1} \ , \tag{58}$$

where we assumed $1 \gg k\tau > k_{\rm mat}\tau \sim R_m\tau$. When $k\tau$ is smaller than $k\tau_m$, $R_m\tau$ term in Eq. (56), which comes from the matter contributions, can not be negligible and should be incorporated appropriately. In the radiation dominant epoch, R_m is represented as

$$R_m \simeq \frac{3}{4}\frac{\Omega_b h^2 + \Omega_c h^2}{\sqrt{\Omega_\gamma h^2 + \Omega_\nu h^2}}\frac{100 \text{ km sec}^{-1} \text{ Mpc}^{-1}}{\text{Mpc}^{-1}} \quad (\text{Mpc}^{-1})$$

$$\simeq 5 \times 10^{-3}\text{Mpc}^{-1}\frac{\Omega_b h^2 + \Omega_c h^2}{0.02 + 0.11}\left(\frac{\Omega_\gamma h^2 + \Omega_\nu h^2}{4.3 \times 10^{-5}}\right)^{-1/2} \ . \tag{59}$$

Then we obtain $k\tau_m$ as shown in Fig. 1. In small scales, $k\tau_m$ become smaller and we can neglect the matter contribution. However, if we start to integrate equations from the conformal time much smaller than τ_m, the matter contribution plays an important role in the magnetic mode. For example, if one sets initial conditions at $k\tau \approx 0.01$, then from Fig. 1 we find that the modes $k \lesssim 0.3$ suffer from the matter contributions. These modes correspond to the angular scale $\ell \lesssim 4000$, at which the difference should be significant as shown in Fig. 1.

B. Numerical Calculation

The equations and initial conditions derived in the previous sections can be used to calculate CMB and matter power spectra numerically. We calculated them with accordingly modified CAMB code [51]. In all of our calculation we fixed cosmological parameters to the best-fitting values to the WMAP-5yr data [52], namely $(\omega_b, \omega_c, h, \tau_c, \Delta_R^2, n_s) = (0.0227, 0.1099, 0.719, 0.087, 2.41\times 10^{-9}, 0.963)$, where ω_b and ω_c are the energy densities of baryon and CDM, respectively, h is the Hubble parameter, τ_c is the optical depth, Δ_R^2 and n_s are the amplitude and the spectral index of primordial curvature fluctuations, respectively.

In Fig. 1, we compare the CMB spectrum of purely magnetic mode calculated with our new initial conditions and that with previous initial conditions, in which the matter contributions were omitted. At higher multipoles, two spectra converge with each other asymptotically. This is because the matter contributions are negligible in very small scales as shown in the previous subsection. However, at large angular scales, we cannot neglect the matter contribution and the correct initial condition yields larger amplitude.

The perturbed Einstein equation gives four equations and two out of four are independent. In order to check the consistency of our numerical calculation, we picked up six kinds of different pairs of two equations from the four independent equations and observed that all of the results coincide with each other. Note that the old initial conditions do not satisfy the Einstein equation. We found that different pairs of perturbed Einstein equations yield different results if we start with the old initial conditions.

In Fig. 2, we plotted the baryon heat flux q_b and curvature perturbation η normalized by the square root of the power spectrum, $\sqrt{\mid \Delta_B \mid^2}$, $\hat{q}_b \equiv q_b/\sqrt{\mid \Delta_B \mid^2}$ and $\hat{\eta} \equiv \eta/\sqrt{\mid \Delta_B \mid^2}$. The characters of the growth of perturbations are different between $k > k_{\rm rec}$ and $k < k_{\rm rec}$, where $k_{\rm rec}$ is a wavenumber which crosses the horizon at the recombination epoch. In the case of $k > k_{\rm rec}$, the perturbation enters the horizon before the recombination. After the horizon-crossing, the growth of the baryon velocity is suppressed and shows oscillatory behavior by the photons pressure through Thomson scatterings. However, after the recombination, q_b grows suddenly by the Lorentz force. This q_b evolution enhances η because the source of the curvature perturbation is the total velocity field, i.e.

$$\eta' = \frac{1}{2k}\kappa\rho a^2 R_f q_f \ . \tag{60}$$

On the other hand, at scales where the waves enter the horizon after recombination, $k < k_{\rm rec}$, the baryon velocity does not undergo the suppression and there is no sudden growth of potentials (dash-dotted (blue) line in the left panel of Fig. 2).

In Fig. 3, two metric perturbations in conformal Newtonian gauge are plotted. Again, we plotted normalized potential, $\hat{\phi}$ and $\hat{\psi}$, with respect to the square root of the power spectrum. Potentials ϕ and ψ are defined as

$$ds^2 = a^2(-(1+2\psi)d\tau^2 + (1-2\phi)dx^2) \ , \tag{61}$$

which is the same definition as in Ref. [40]. The relations with the variables used in this paper and their initial conditions are given as

$$\begin{aligned} \phi &= \eta - \frac{1}{k}\mathcal{H}\sigma = -3R_\nu \pi^{(2)} + \frac{R_m}{8k}R_\gamma \Delta_B k\tau \ \ , \\ \psi &= \frac{1}{k}(\sigma' + \mathcal{H}\sigma) = -6R_\nu \pi^{(2)} + \frac{R_m}{8k}R_\gamma \Delta_B k\tau \ \ . \end{aligned} \tag{62}$$

In the absence of the magnetic field, two potentials decay to zero, once the perturbation enter the horizon in radiation dominant epoch (thick lines in Figs. 2 and 3), and are constant in time in matter dominated epoch. However, in the purely magnetic mode, potentials grow after the recombination.

Next we study the growth of curvature perturbations in the adiabatic mode correlated with the magnetic mode. In Fig. 4, the growths of η in full and anti correlated cases are plotted. If the adiabatic mode is fully correlated with the magnetic mode, η grows after the recombination for $k > k_{\rm rec}$, while η decays in the anti correlated case. This growth of the curvature perturbation is directly seen in the CMB and matter power spectra as shown in Fig. 5. For the CMB spectrum (three panels in Fig. 5), primary standard adiabatic mode and uncorrelated mode have a similar feature in shape and almost degenerated. Therefore, if the adiabatic mode is fully correlated with the magnetic mode, the gravitational potential becomes deeper and the amplitudes of spectra are increased. On the other hand, the anti correlated magnetic mode decays the potential and amplitudes of spectra become smaller.

For the matter power spectrum, on the other hand, the difference shows up at small scales because the Lorentz force from magnetic fields newly induces the density perturbations dominantly at small scales after cosmological recombination. For both correlation cases with 300 nG magnetic field, the linear power at $k \gtrsim 1\mathrm{Mpc}^{-1}$ is dominated by the perturbations induced by magnetic fields.

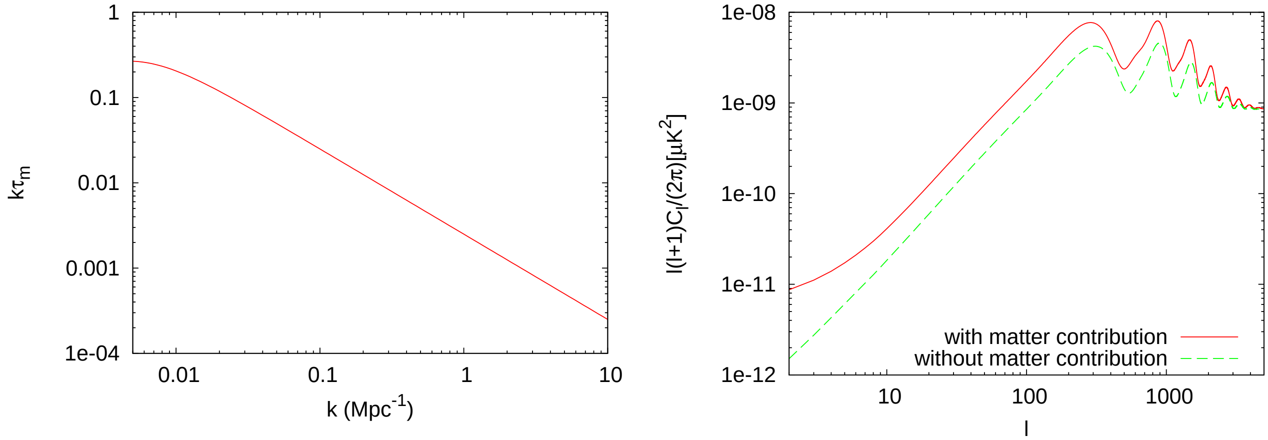

FIG. 1: Left: The conformal time at which the matter contribution becomes significant, $k\tau_m$. If $k\tau < k\tau_m$, the matter contribution can not be neglected. Right: Comparison of the CMB power spectra derived from the initial condition with matter contribution and from the one without. The new initial condition leads the larger amplitude. Magnetic field parameters are $B_\lambda = 1\text{nG}$ and $n_B = -2.5$.

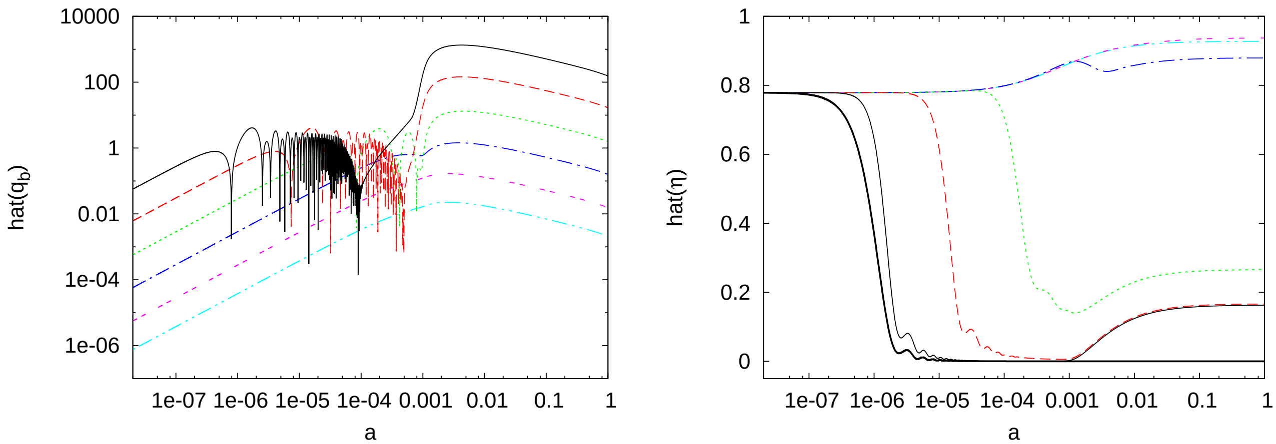

FIG. 2: The normalized baryon heat flux $\hat{q}_b$ (left) and curvature perturbation $\hat{\eta}$ (right) of the purely magnetic mode. The baryon velocity evolves by the Lorentz force after recombination. This enhances the curvature potential at small scales. Each thin line represents the purely magnetic mode with different scale: solid (black), dashed (red), dotted (green), dash-dotted (blue), double dotted (magenta), and dashed-double dotted (light blue) lines corresponds to the modes with $k = 10$, 1, 10^{-1}, 10^{-2}, 10^{-3}, and 10^{-4} Mpc^{-1}, respectively. In the right panel, a thick solid line represents the standard adiabatic mode with $k = 10\text{Mpc}^{-1}$, which decays to zero in the radiation dominated era and stays constant in the matter dominated era.

V. CONCLUSION

In this work, we derived the initial conditions for density perturbations in the existence of the primordial magnetic field. Because the compensation mechanism between radiation perturbation and the magnetic field makes it impossible to neglect matter contributions even in the early universe, we derived initial conditions including matter contributions to the density field. The initial condition derived in this paper fully satisfies the linearized Einstein equations, and thus enables us to solve the system numerically in a stable and consistent manner. Then CMB and matter power spectra are calculated, and the evolutions of perturbations are presented in detail. We found that it gives the larger amplitude of the CMB angular power spectrum by at most a order of magnitude at large scales $l \lesssim 4000$, compared with the initial condition in the literature.

In the purely magnetic mode, the potentials grow suddenly after recombination for $k > k_{\text{rec}}$. This effect enhance the amplitude of the matter power spectrum if the adiabatic and magnetic mode are fully correlated, and decreases the amplitude in the anti correlated case at intermediate scales. For much smaller scales $k \gtrsim 1$ Mpc^{-1}, the spectrum is dominated by the magnetic mode.

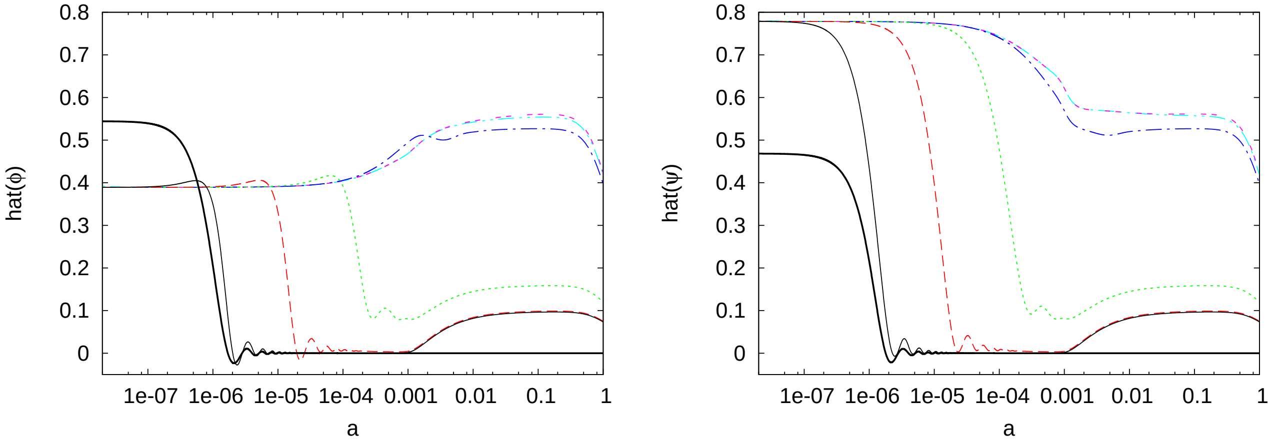

FIG. 3: Gravitational potentials $\hat{\phi}$ and $\hat{\psi}$ of the purely magnetic mode in conformal Newtonian gauge. Potentials grows suddenly at small scales after the recombination. Each thin line represents the purely magnetic mode with different scale: solid (black), dashed (red), dotted (green), dash-dotted (blue), double dotted (magenta), and dash-double dotted (light blue) lines corresponds to the mode with $k = 10$, 1, 10^{-1}, 10^{-2}, 10^{-3}, and 10^{-4} Mpc^{-1}, respectively. For both panels, thick solid lines are the standard adiabatic modes with $k = 10\mathrm{Mpc}^{-1}$.

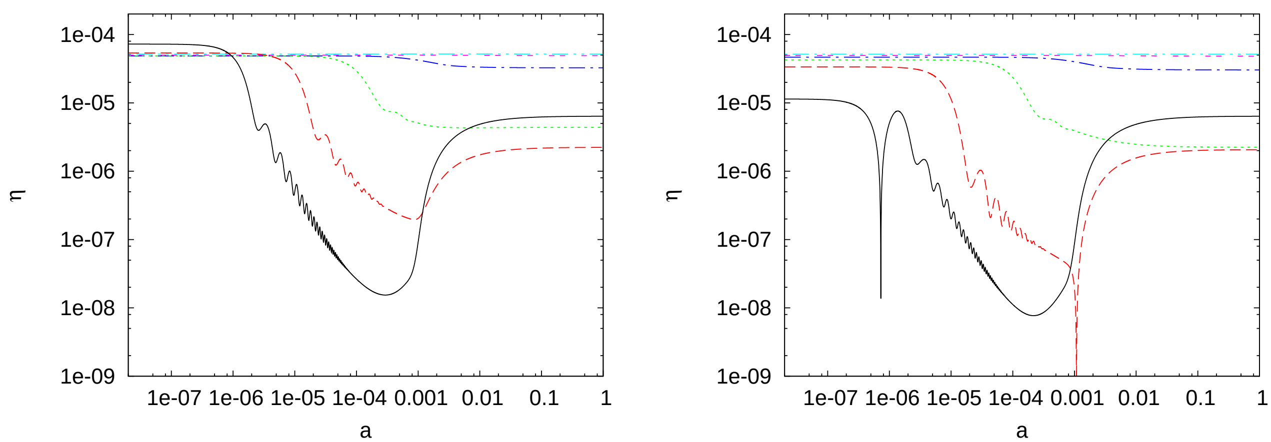

FIG. 4: The growths of η in fully (left) and anti (right) correlated cases. The magnetic field enhances the curvature perturbation in the fully correlated case. On the other hand, anti correlated magnetic field diminishes η. Each thin line represents different scale: solid (black), dashed (red), dotted (green), dash-dotted (blue), double dotted (magenta), and dash-double dotted (light blue) lines corresponds to the mode with $k = 10$, 1, 10^{-1}, 10^{-2}, 10^{-3}, and 10^{-4} Mpc^{-1}, respectively. Parameters for magnetic fields are $B_\lambda = 300\mathrm{nG}$ and $n_B = -2.5$.

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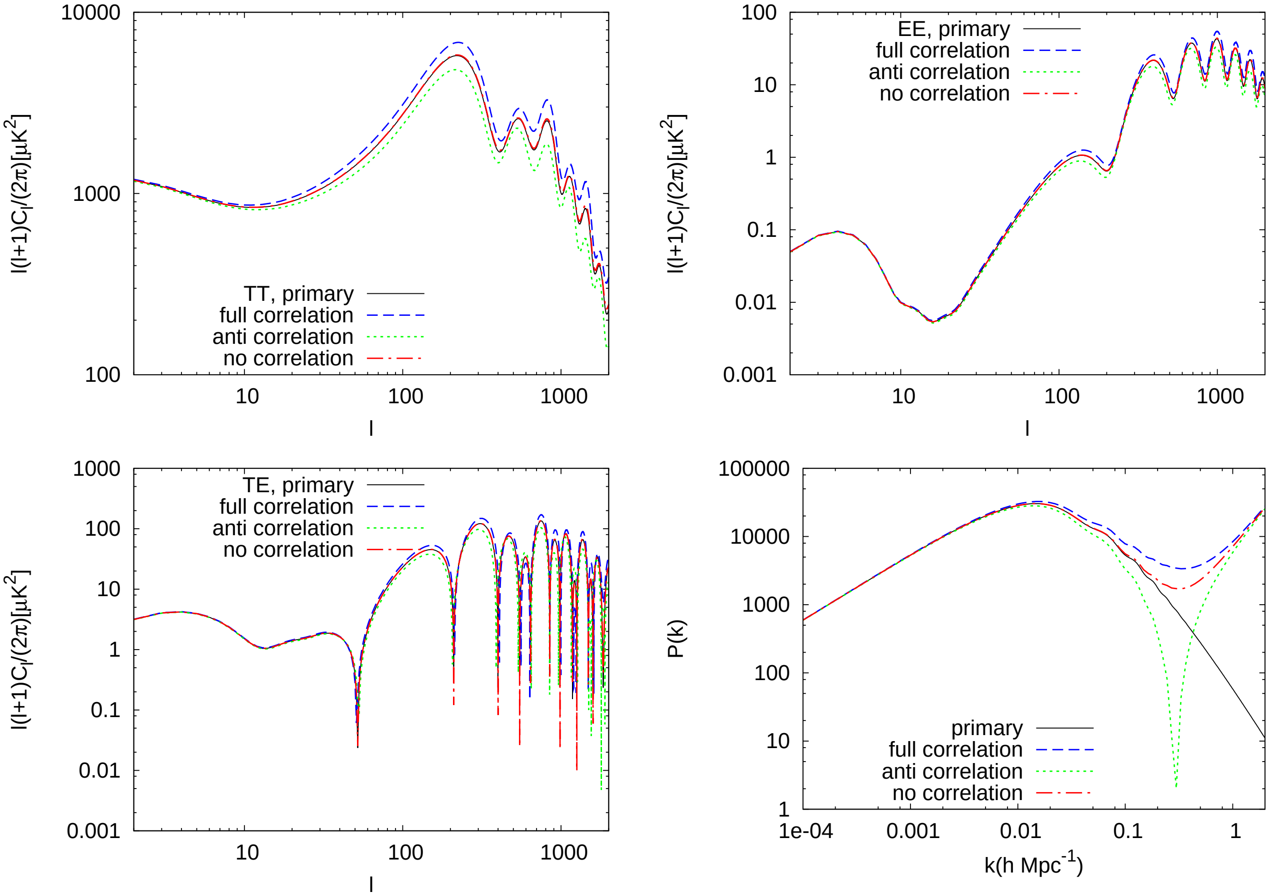

FIG. 5: The CMB spectra of TT, EE and TE mode and the matter power spectrum. If the magnetic fields of 300 nG are uncorrelated with the primary adiabatic modes the effects are not significant. However, if the adiabatic mode is fully correlated with the magnetic field, the gravitational potential grows and enhances the spectra, which can be seen in the figures. On the other hand, the amplitude decreases in the anti correlated case. Parameters for magnetic fields are $B_\lambda = 300\text{nG}$ and $n_B = -2.5$.

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